HORNSBY GIRLS' HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- **Reading Time – 5 minutes**
- **Working Time – 3 hours**
- **Write using a black or blue pen**
- **Approved calculators may be used**
- **A table of standard integrals is provided at the back of this paper.**
- **All necessary working should be shown for every question.**
- **Begin each question on a fresh sheet of paper.**

Total marks (120)

- **Attempt Questions 1 – 10**
- **All questions are of equal value**

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Total Marks – 120

Attempt Questions 1-10
All Questions are of equal value

Answer each question in a SEPARATE writing booklet, writing your student number and question number on the cover of each booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\dfrac{3.9\times 15.6}{12.5-4.8^3}$ correct to 3 significant figures. **2**

(b) Find the exact value of $\log_3 \sqrt{27}$ **2**

(c) Solve $5-(1+x)\geq 0$ **2**

(d) Find integers a and b such that $\dfrac{7}{5+3\sqrt{2}} = a+b\sqrt{2}$ **2**

(e) Solve the pair of simultaneous equations **2**

$$3x+y=5$$
$$x-2y=4$$

(f) A sector of a circle with radius 9cm has an arc length of 6π cm. **2**
Calculate the size of the angle subtended at the centre.
Give your answer in radians.

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

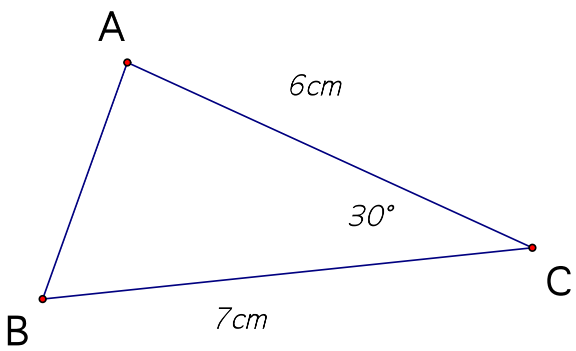
(a) Find : (i) $\displaystyle\int \frac{x+1}{x^2}\,dx$ **2**

(ii) $\displaystyle\int 10\sec^2 2x\ dx$ **2**

(b) The graph of $y = f(x)$ passes through the point $(-1, 4)$ and $f'(x) = 5 - 3x^2$. **2**
Find $f(x)$.

(c) Find the gradient of the tangent to the curve $y = \sqrt{x+2}$ at the point where $x = 7$. **2**

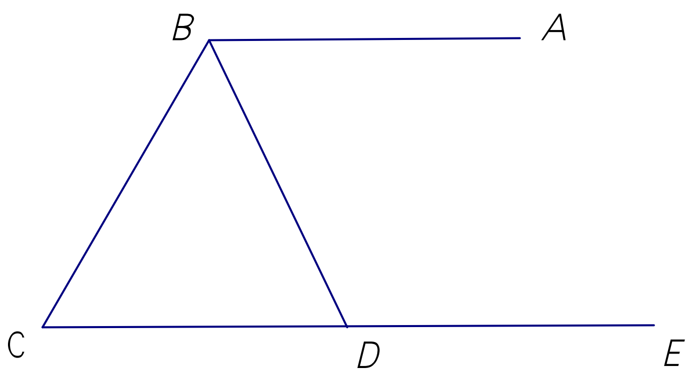
(d) In the diagram, $AC = 6cm$, $BC = 7cm$ and $\angle ACB = 30°$.



NOT TO SCALE

Find the length of AB correct to the nearest centimetre. **2**

(e)



In the diagram, $BC = BD$, reflex $\angle ABC = 220°$, $\angle BCD = 40°$. **2**
Prove that $AB \parallel EC$.

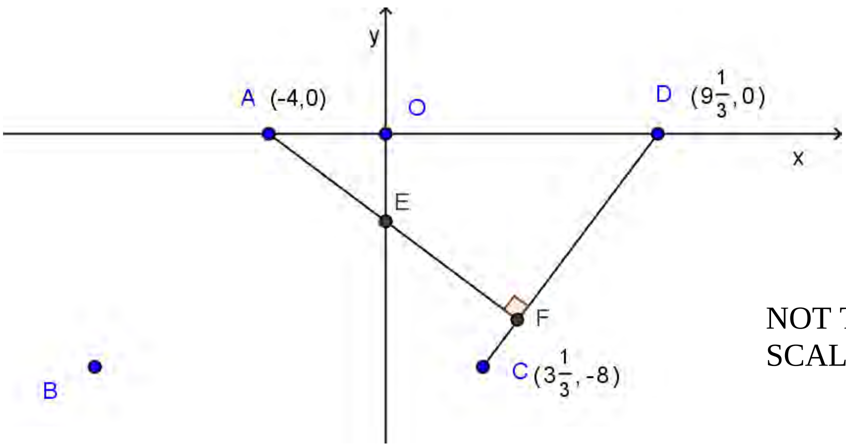
Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the derivative of the following functions, in simplest form:

(i) xe^x **2**

(ii) $\dfrac{-\cos x}{x^2}$ **2**

(b)



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The diagram shows the points $A(-4,\ 0)$, $C(3\frac{1}{3},\ -8)$ and $D(9\frac{1}{3},0)$.

(i) If ABCD is a parallelogram, what are the coordinates of B? **1**

(ii) Find the equation of CD in general form. **2**

(iii) Show that the exact perpendicular distance of A from CD is $10\frac{2}{3}$ units. **2**

(iv) Prove ΔAOE and ΔAFD are similar. **2**

(v) Hence, or otherwise, find AE. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

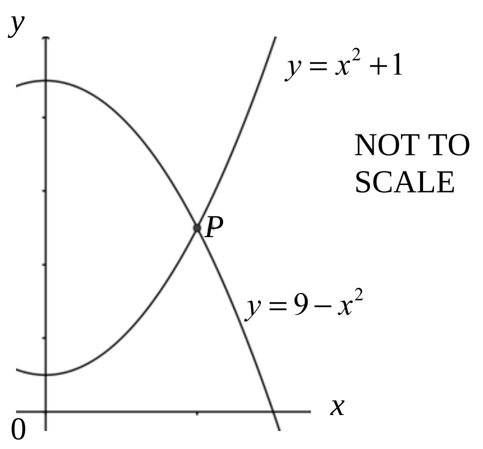
(a) Find the values of k for which $x^2 + (k+6)x - 2k = 0$ has real roots. **2**

(b) For the last phase of preparations before the Olympic Games, swimmers started a new training schedule.
On the first day they had to complete 26 laps of the pool.
Each succeeding day they increased their training by 6 laps, until their daily schedule reached 200 laps. They then continued swimming 200 laps daily for a total of 15 days to fully complete their training schedule.
(Note: length of pool = 50m)

(i) On which day did they first complete 200 laps? **2**

(ii) Find the total distance (in kilometres) completed by the swimmers. **3**

(c)



The diagram shows the parts of the graphs $y = x^2 + 1$ and $y = 9 - x^2$ in the positive quadrant.

(i) Show that the co-ordinates of the point P are (2, 5). **1**

(ii) Copy the diagram into your examination booklet and shade the region represented by the inequalities: **1**
$x \geq 0$, $y \geq 0$, $y \leq x^2 + 1$ and $y \leq 9 - x^2$.

(iii) Calculate the area of this shaded region. **3**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the curve given by $y = x^3 + 3x^2 - 9x$ find:

(i) $\frac{dy}{dx}$ **1**

(ii) the coordinates of the stationary points and determine their nature. **3**

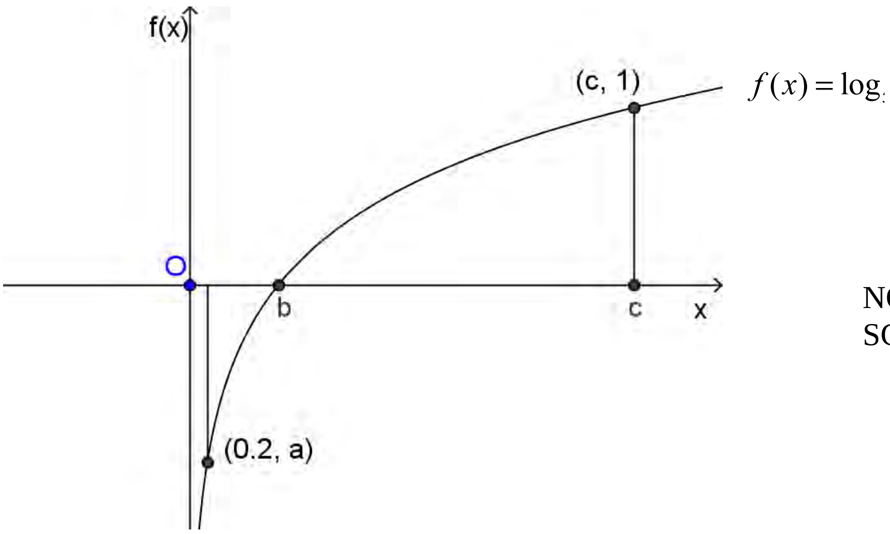
(iii) the point of inflexion. **1**

(iv) the set of values of x for which the graph is concave downwards. **1**

(v) the maximum value of y for $-6 \le x \le 4$. **1**

(b) Solve: $2\log_4 x + \log_4 \sqrt{x} = \log_4 32$ **2**

(c)



NOT TO SCALE

For the graph above:

(i) Is the following statement correct for the function shown? Explain why or why not.

$$\int_{0.2}^{c} f(x)dx = \int_{0.2}^{b} f(x)dx + \int_{b}^{c} f(x)\,dx$$ **1**

(ii) Find the values of the pronumerals a, b and c. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $2\cos^2\theta - 3\cos\theta + 1 = 0$ for $0 \le \theta \le 2\pi$. **3**

(b) The amount of water in litres, remaining in a storm water tank, after *t* minutes, is given by $V = 10000e^{kt}$. After 60 minutes, 8000L remains in the tank.

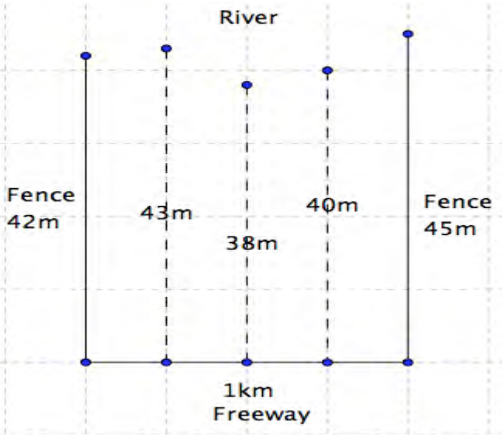
(i) Find the initial amount of water in the storm water tank. **1**

(ii) Find the value of *k*. **2**

(iii) How much water would remain after 120 minutes? **2**

(iv) Will the tank ever be empty? Give reasons for your answer. **1**

(c) The below diagram shows a field which is bounded by a river, a highway and two fences.



Use Simpson's rule with 5 function values to approximate the area of the field. **3**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In an opinion poll conducted at a school, it was found that 70% of students were in favour of a uniform **without** a tie. Three students were selected at random. Find the probability that:

(i) all three students favoured a uniform without a tie. **2**

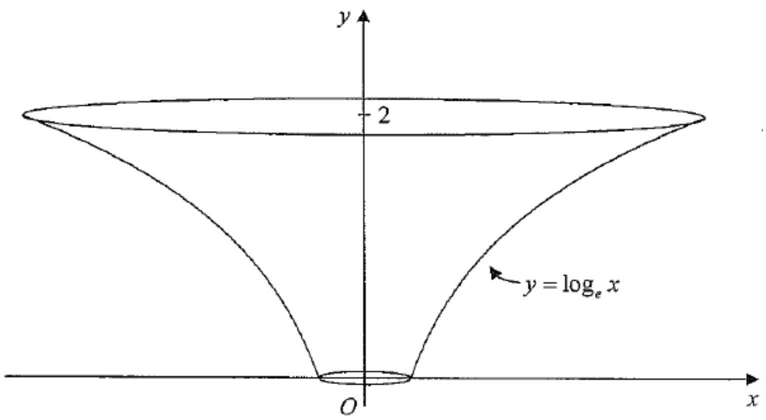
(ii) at least one student favoured a uniform without a tie. **2**

(b) The sum of the radii of two circles is constant.
HINT: Let the radii of the circles be R , r whereby $R + r = c$, a constant.

(i) Show that the sum of the areas, S, is given by $S = \pi(2r^2 - 2cr + c^2)$. **2**

(ii) Prove that the sum of the areas of the two circles is least when the circles have equal radii. **3**

(c) A mould for a vase is formed by rotating that part of the curve $y = \log_e x$ between $y = 0$ and $y = 2$ about the y-axis. **3**



Find the volume of the mould. Leave your answer in simplest exact form.

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A parabola has focus (3, 2) and directrix $x = -1$.

(i) Sketch the parabola showing these features and its vertex and axis. **2**

(ii) State the equation of the parabola. **1**

(b) A particle P moves along the x-axis. The velocity v, in cm/s, is given by the equation $v = 1 - 2\sin t,\ t \geq 0$, where t is the time in seconds.
Initially the particle is 2cm to the right of the origin.

(i) Find an expression for the position of the particle at time *t*. **2**

(ii) When does the particle change direction in the first π seconds? **2**

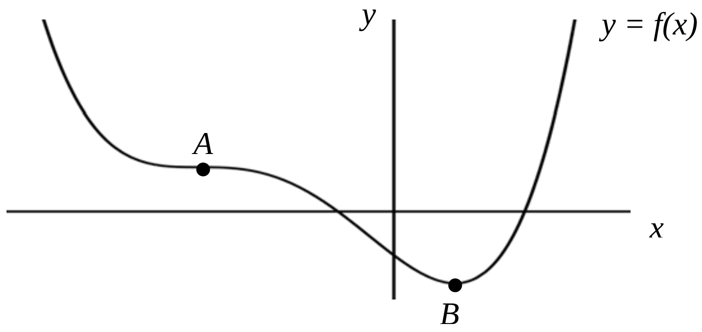
(iii) Determine the exact value of the distance travelled during the first $\dfrac{\pi}{2}$ seconds. **2**

(iv) Particle Q moves along the x-axis so that its position is given by the equation $x = 6 + t + 2\cos t,\ t \geq 0$. **1**

Describe the motion of Q relative to the particle P.

(c) Below is the graph of $y = f(x)$ which has a horizontal point of inflection at *A* and a minimum turning point at *B*. **2**

Copy this diagram onto your answer sheet and sketch the graph of its derivative on the same axes.



Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $f(x) = 3\sin \pi x$,

(i) find the period and amplitude of the function. **2**

(ii) sketch the curve $y = f(x)$ for $0 \le x \le 4$. **1**

(iii) determine the number of solutions of the equation $3\sin \pi x = x$ in the interval $0 \le x \le 4$. **2**

(b) In a small town, the number of residents who caught a disease t weeks after a virus was introduced can be represented by:

$$N(t) = 1000\left(1 - \frac{1}{1+t^2}\right), \text{ for } t \ge 0.$$

(i) How many residents are in the town? **1**

(ii) Calculate the number of residents who caught the disease after one week. **1**

(iii) After what length of time, to the nearest day, had 75% of the residents caught the disease? **2**

(iv) At what rate was the disease spreading three weeks after the disease was introduced? **3**

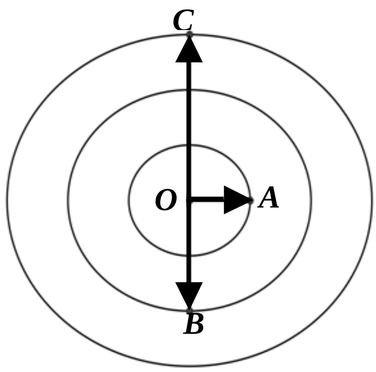
Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the geometric series $54 - 18a + 6a^2 - 2a^3 + \ldots$.

(i) For what values of a does the series have a limiting sum? **2**

(ii) The limiting sum of series is 81. Find the value of a. **2**

(b)



In a modified sport of archery, the scoring system is as follows:
the bull's eye (i.e. innermost circle with radius *OA* = 5 cm) is worth 10 points,
next circle with radius *OB* =10 cm is worth 5 points,
the largest circle with radius *OC* = 15 cm is worth 1 point and outside the circles scores 0 points.

(i) If a player takes one shot at the target board, what is the probability that he scores 10 points? **1**

(ii) If the same player has three shots at the target and hits it each time, what is the probability that he scores a total of 20 points? **2**

(c) Veronica opened a perfume factory on January 1, 2000.
Her initial stock was 60 000 litres of "Essence" base material.
During any month she used 1% of the stock existing at the beginning of that month and to maintain a sufficient level of stock, she purchased an additional 100 litres "Essence" on the last day of each month.

(i) Write down an expression for A_1, the number of litres of "Essence" in stock after one month. **1**

(ii) Show that A_n, the number of litres of "Essence" in stock after n months, is given by

$$A_n = 50\ 000 \times 0.99^n + 10\ 000$$ **2**

(iii) During which month and year will her stock first fall below 90% of her initial stock? **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

TRIAL 2009

a) $\dfrac{3 \cdot 9 \times 15 \cdot 6}{12 \cdot 5 - 4 \cdot 8^3} = -0.620234066$

$= -0.620$ (3 sig figs)

b) $\log_3 \sqrt{27} = \log_3 27^{\frac{1}{2}}$

$= \log_3 (3^3)^{\frac{1}{2}}$

$= \log_3 3^{3/2}$

$= \frac{3}{2}$

c) $5-(1+x) \geq 0$

$5 - 1 - x \geq 0$

$4 - x \geq 0$

$-x \geq -4$

$x \leq 4$

d) $\dfrac{7}{5+3\sqrt{2}} = \dfrac{7}{5+3\sqrt{2}} \times \dfrac{5-3\sqrt{2}}{5-3\sqrt{2}}$

$= \dfrac{7(5-3\sqrt{2})}{25-(9\times 2)}$

$= \dfrac{35-21\sqrt{2}}{7}$

$= 5-3\sqrt{2}$

$\therefore a=5, \quad b=-3$

e) $3x+y=5 \quad \ldots(1)$

$x-2y=4 \quad \ldots(2)$

(1) ×2 $\quad 6x+2y=10 \quad \ldots(3)$

$x-2y=4 \quad \ldots(2)$

(3) + (2) $\quad 7x=14$

$x=2$

sub $x=2$ into (1)

$6+y=5$

$y=-1$

Solution: $x=2, \; y=-1$

f) 

$l = r\theta$

$6\pi = 9\theta$

$\theta = \dfrac{6\pi}{9}$

$\theta = \dfrac{2\pi}{3}$ radians.

QUESTION 2

a) (i) $\displaystyle\int \frac{x+1}{x^2}\,dx$

$= \displaystyle\int \frac{1}{x} + \frac{1}{x^2}\,dx$

$= \displaystyle\int \frac{1}{x} + x^{-2}\,dx$

$= \ln x + \dfrac{x^{-1}}{-1} + c$

$= \ln x - \dfrac{1}{x} + c$

(ii) $\displaystyle\int 10\sec^2 2x\,dx$

$= 10 \times \frac{1}{2}\tan 2x + c$

$= 5\tan 2x + c$

b) $f'(x) = 5-3x^2$

$f(x) = 5x - x^3 + c$

But curve passes through $(-1,4)$

$\therefore 4 = -5+1+c$

$8 = c$

$\therefore f(x) = 5x - x^3 + 8$

c) $y = \sqrt{x+2}$

$= (x+2)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \frac{1}{2}(x+2)^{-\frac{1}{2}} \times 1$

$= \dfrac{1}{2\sqrt{x+2}}$

when $x=7$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{9}}$

$= \dfrac{1}{6}$

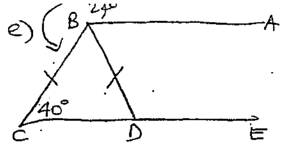
$\therefore$ gradient of tangent $= \frac{1}{6}$.

d) $AB^2 = 6^2+7^2 - 2\times 6 \times 7 \times \cos 30°$

$= 12.25386608\ldots$

$AB = 3.50055\ldots$

$AB = 4$ cm (to the nearest cm).

e) 

In $\triangle BCD$, $BC = BD$

$\therefore \triangle BCD$ is isosceles

$\therefore \angle BDC = 40°$ (equal ∠'s of an isosceles △)

$\therefore \angle CBD = 100°$ (angle sum of $\triangle BCD$)

$\therefore \angle ABD = 360° - (220° + 100°)$ (angles at a point)

$= 40°$

$\therefore \angle ABD = \angle BDC$ (both 40°)

$\therefore AB \parallel EC$ (alternate angles are equal)

QUESTION 3

a) (i) $y = xe^x$

$\dfrac{dy}{dx} = uv' + vu'$

$= xe^x + e^x.1$

$= e^x(x+1)$

(ii) $y = \dfrac{-\cos x}{x^2}$

$\dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{x^2 \sin x + \cos x . 2x}{x^4}$

$= \dfrac{x(x\sin x + 2\cos x)}{x^4}$

$= \dfrac{x\sin x + 2\cos x}{x^3}$

b) (i) $m_{CD} = \frac{8}{6}$

$\therefore B(-4-6, \; 0-8) = (-10,-8)$

(ii) $m_{CD} = \frac{4}{3} \quad D(9\frac{1}{3}, 0)$

Equation of CD:

$y - 0 = \frac{4}{3}(x - 9\frac{1}{3})$

$3y = 4(x-9\frac{1}{3})$

$9y = 12(x - 9\frac{1}{3})$

$9y = 12x - 112$

$12x - 9y - 112 = 0$

(iii) $A(4,0) \quad CD: 12x - 9y - 112 = 0$

$d = \left| \dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$

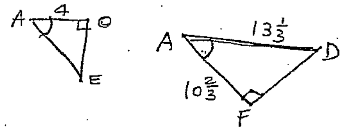
$= \left| \dfrac{12(4) - 9(0) - 112}{\sqrt{144+81}} \right|$

$= \left| \dfrac{-48-112}{\sqrt{225}} \right|$

$= \left| \dfrac{-160}{15} \right|$

$= 10\frac{2}{3}$ units.

(iv) In △'s AOE and AFD



$\angle OAE = \angle DAF$ (common angle)

$\angle AOE = \angle AFD$ (both 90°)

$\therefore \angle AEO = \angle ADF$ (angle sums of △'s)

$\therefore \triangle ADE \;|||\; \triangle AFD$ (equiangular)

(v) Since $\triangle AOE \;|||\; \triangle AFD$, corresponding sides are in the same ratio

$\therefore \dfrac{AE}{AD} = \dfrac{AO}{AF}$

$\dfrac{AE}{13\frac{1}{3}} = \dfrac{4}{10\frac{2}{3}}$

$AE = \dfrac{4}{10\frac{2}{3}} \times 13\frac{1}{3}$

[OR $\dfrac{AE}{40} = \dfrac{4}{32}$]

$AE = \dfrac{160}{32}$

$AE = 5$ units.

QUESTION

a) $\Delta = b^2 - 4ac$
$= (k+6)^2 - 4(-2k)$
$= k^2 + 12k + 36 + 8k$
$= k^2 + 20k + 36$

For real roots $\Delta \geq 0$
$k^2 + 20k + 36 \geq 0$
$(k+18)(k+2) \geq 0$
$k \leq -18$ or $k \geq -2$

b) $26 + 32 + 38 + 44 + \dots + 200$

(i) Arithmetic series $a = 26$
$d = 6$
$T_n = 200$

$a + (n-1)d = 200$
$26 + 6(n-1) = 200$
$26 + 6n - 6 = 200$
$6n = 180$
$n = 30$

They completed 200 laps on the 30^{th} day

(ii) Total laps for first 30 days

$= \frac{n}{2}[a + l]$
$= \frac{30}{2}[26 + 200]$
$= 15(226)$
$= 3390$

Total laps for next 15 days
$= 15 \times 200$
$= 3000$ laps.

Total distance $= 6390 \times 50$m
$= 319\,500$m
$= 319.5$ km

c) (i) P is the point of intersection of $y = x^2 + 1$ and $y = 9 - x^2$

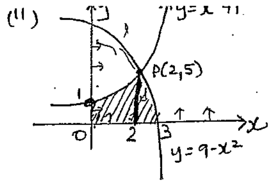
let $x^2 + 1 = 9 - x^2$
$2x^2 = 8$
$x^2 = 4$
$x = \pm 2$

But P is in the positive quadrant $\therefore x = 2$

when $x = 2$, $y = (2)^2 + 1$
$= 5$

$\therefore$ P is the point $(2,5)$

(ii)



(iii) $y = 9 - x^2$: x-intercepts – let $y = 0$
$9 - x^2 = 0$
$x = \pm 3$
In positive quad, $x = 3$

$\therefore A = \int_0^2 (x^2+1)\,dx + \int_2^3 (9 - x^2)\,dx$

$= \left[\frac{x^3}{3} + x\right]_0^2 + \left[9x - \frac{x^3}{3}\right]_2^3$

$= \left[\left(\frac{8}{3} + 2\right) - (0)\right] + \left[(27 - 9) - \left(18 - \frac{8}{3}\right)\right]$

$= \left(4\frac{2}{3}\right) + \left(2\frac{2}{3}\right)$

$= 7\frac{1}{3}$ unit^2

a) $y = x^3 + 3x^2 - 9x$

(i) $\frac{dy}{dx} = 3x^2 + 6x - 9$

(ii) At stationary points $\frac{dy}{dx} = 0$
$3x^2 + 6x - 9 = 0$
$3(x^2 + 2x - 3) = 0$
$3(x+3)(x-1) = 0$
$x = -3$ or $+1$

when $x = -3$, $y = (-3)^3 + 3(-3)^2 - 9(-3)$
$= 27$ $\quad(-3, 27)$

when $x = +1$, $y = 1 + 3 - 9$
$= -5$ $\quad(1, -5)$.

$f''(x) = 6x + 6$
$f''(-3) = -18 + 6 < 0$
$\therefore$ Max turning pt at $(-3, 27)$
$f''(1) = 6 + 6 > 0$
$\therefore$ Min turning pt at $(1, -5)$

(iii) Possible points of inflexion $f''(x) = 0$
$6x + 6 = 0$
$x = -1$

$f''(-2) = -12 + 6 < 0$ concave down
$f''(0) = 0 + 6 > 0$ concave up

$\therefore$ since concavity changes, there is a point of inflexion at $(-1, 11)$

$y = (-1)^3 + 3(-1)^2 - 9(-1)$
$= -1 + 3 + 9$
$= 11$

(iv) Concave down $\Rightarrow f''(x) < 0$
$6x + 6 < 0$
$x < -1$

(v) Endpoints: $f(-6) = -54$
$f(4) = 76$
Max t.p. $(-3, 27)$
$\therefore$ Maximum value $= 76$

b) $2\log_4 x + \log_4 \sqrt{x} = \log_4 32$

$\log_4 x^2 + \log_4 x^{\frac{1}{2}} = \log_4 32$

$\log_4 (x^2 \times x^{\frac{1}{2}}) = \log_4 32$

$x^{5/2} = 32$

$x = (32)^{\frac{2}{5}}$

$x = 4$

c) (i) The statement is correct as the question is not an area question so we do not need to add the absolute values

Both left hand side and right hand side would be adding a negative value to a positive value to achieve the same answer

Alternative Solution for (c)(i)

c)(i) Let area between curve and x axis from $x = 0.2$ to $x = b$ be 2 units

Let area between curve and y-axis from $x = b$ to $x = c$ be 'y' units

LHS $= \int_{0.2}^{c} f(x)\,dx$
$= -2 + y$
$= y - 2$

RHS $= \int_{0.2}^{b} f(x)\,dx + \int_{c}^{b} f(x)\,dx$
$= -2 + y$
$= y - 2$ units

(ii) $b = 1$

$a = \log_5 0.2$
$= \log_5 \left(\frac{1}{5}\right)$
$= \log_5 5^{-1}$
$= -1$

$\log_5 c = 1$
$c = 5^1$
$c = 5$

a) $2\cos^2\theta - 3\cos\theta + 1 = 0$

$(2\cos\theta - 1)(\cos\theta - 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 1$

$\theta = \frac{\pi}{3}, (2\pi - \frac{\pi}{3})$ $\quad \theta = 0, 2\pi$

$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}, 0, 2\pi$

b) $V = 10000e^{kt}$

(i) when $t=0$, $V = 1000e^0 = 1000$

∴ Initially there is 10000 L of water in the tank

(ii) when $t=60$ $V = 8000L$

$\therefore 8000 = 10000e^{60k}$

$0.8 = e^{60k}$

$60k = \ln 0.8$

$k = \frac{\ln 0.8}{60}$

(iii) when $t=120$

$V = 10000e^{\frac{\ln 0.8}{60} \times 120}$

$= 10000e^{2\ln 0.8}$

$= 6400$ L

∴ Volume remaining after 120 minutes is 6400L

(iv) Let $V=0$

$1000e^{kt} = 0$

$e^{kt} = 0$

But $e^{kt} > 0$ ∴ No solution

∴ The tank will never be empty.

$h = \frac{1000m}{4} = 250$m. $\quad$ 5 function values

$A \doteqdot \frac{h}{3}[y_1 + 4y_2 + 2y_3 + 4y_4 + y_5]$

$= \frac{250}{3}[42 + 4(43+40) + 2(38) + 45]$

$= 41250\ m^2$

a) (i) $P(\bar{T}\bar{T}\bar{T}) = 0.7 \times 0.7 \times 0.7 = 0.343$

(ii) P(at least one student favoured uniform without tie)

= 1 − P(none favoured uniform without tie)

$= 1 - P(TTT)$

$= 1 - (0.3 \times 0.3 \times 0.3)$

$= 1 - 0.027$

b) $R + r = c$

$\therefore R = c - r$

(i) $S = \pi R^2 + \pi r^2$

$= \pi(c-r)^2 + \pi r^2$

$= \pi(c^2 - 2cr + r^2) + \pi r^2$

$= \pi(c^2 - 2cr + r^2 + r^2)$

$= \pi(2r^2 - 2cr + c^2)$

(ii) $\frac{dS}{dr} = \pi(4r - 2c)$

For greatest/least sum - let $\frac{dS}{dr} = 0$

$\pi(4r - 2c) = 0$

$4r = 2c$

$r = \frac{2c}{4}$

$r = \frac{1}{2}c$

$\frac{d^2S}{dr^2} = \pi(4) > 0$ (Concave up) MINIMUM

∴ least sum occurs when $r = \frac{1}{2}c$

$\therefore R = c - r = c - \frac{1}{2}c = \frac{1}{2}c$

∴ Sum of the areas is least when the radii are equal.

c) $y = \log_e x$

$x = e^y$

$V = \pi\int_0^2 x^2\,dy$

$= \pi\int_0^2 (e^y)^2\,dy$

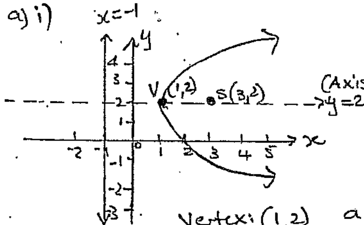
$= \pi\int_0^2 e^{2y}\,dy$

$= \pi\left[\frac{1}{2}e^{2y}\right]_0^2$

$= \frac{\pi}{2}[e^4 - e^0]$

$= \frac{\pi}{2}(e^4 - 1)$ unit³

a) i)



Vertex: (1,2) $\quad a=2$

Axis: $y=2$

ii) $(y-2)^2 = 8(x-1)$

b) $v = 1 - 2\sin t$, $t \geq 0$ $\quad$ v cm/s, t s

when $t=0$, $x=2$

(i) $x = \int 1 - 2\sin t\,dt$

$= t + 2\cos t + c$

when $t=0$, $x=2$

$\therefore 2 = 0 + 2\cos 0 + c$

$2 = 2 + c$

$c = 0$

$\therefore x = t + 2\cos t$

(ii) Particle changes direction when $v=0$

Let $1 - 2\sin t = 0$

$\sin t = \frac{1}{2}$

$t = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

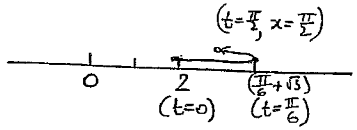
∴ The particle changes direction after $\pi/6$ seconds and $\frac{5\pi}{6}$ seconds.

(iii) when $t=0$, $x = 0 + 2\cos 0 = 2$

when $t = \frac{\pi}{6}$, $x = \frac{\pi}{6} + 2\cos\frac{\pi}{6} = \frac{\pi}{6} + \sqrt{3}$

when $t = \frac{\pi}{2}$, $x = \frac{\pi}{2} + 2\cos\frac{\pi}{2} = \frac{\pi}{2}$

∴



Distance travelled $= \left[\left(\frac{\pi}{6} + \sqrt{3}\right) - 2\right] + \left[\left(\frac{\pi}{6} + \sqrt{3}\right) - \frac{\pi}{2}\right]$

$= \frac{2\pi}{6} + 2\sqrt{3} - 2 - \frac{\pi}{2}$

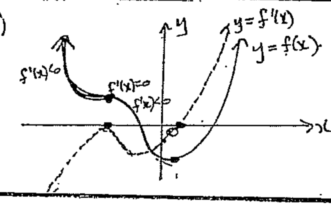
$= \left(\frac{\pi}{3} + 2\sqrt{3} - 2 - \frac{\pi}{2}\right)$ cm.

(iv) Particle P: $x = t + 2\cos t$, $v = 1 - 2\sin t$

Particle Q: $x = 6 + (t + 2\cos t)$, $v = 1 - 2\sin t$.

∴ Particle Q travels at the same speed as particle P but is always 6 cm ahead of particle P.

c)

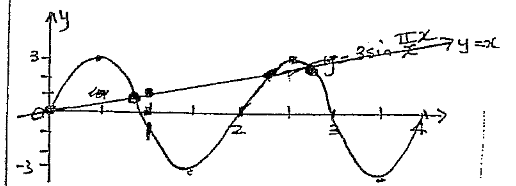


QUESTION 9

a) $f(x) = 3\sin\pi x$

(i) period $= \frac{2\pi}{\pi} = 2$ $\quad$ amplitude $= 3$

(ii) 1 wave in 2 units
2 waves in 4 units



(iii) $y = 3\sin\pi x$, $y = x$ } $3\sin\pi x = x$

The solutions will be the x-values of the points of intersection

∴ There will be 4 solutions

b) $N(t) = 1000\left(1 - \frac{1}{1+t^2}\right), \; t \geq 0.$

(i) As $t \to \infty$, $N(t) \to 1000$

Initially there were 1000 residents in the town.

(ii) When $t=1$,

$N(t) = 1000\left(1 - \frac{1}{1+1}\right)$

$= 1000 \times \frac{1}{2}$

$= 500$

After one week, 500 residents caught the disease

i) 75% of 1000 = 750

$750 = 1000\left(1 - \frac{1}{1+t^2}\right)$

$0.75 = 1 - \frac{1}{1+t^2}$

$\frac{1}{1+t^2} = 0.25$

$\frac{1}{1+t^2} = \frac{1}{4}$

$1+t^2 = 4$

$t^2 = 3$

$t = \pm\sqrt{3}$

But $t \geq 0$, $\therefore t = \sqrt{3}$

$\doteqdot 1.73205\ldots$ weeks

$\doteqdot 12$ days (to nearest day)

i) Rate $= \frac{dN}{dt}$

$N = 1000\left(1 - \frac{1}{1+t^2}\right)$

$= 1000\left[1 - (1+t^2)^{-1}\right]$

$\frac{dN}{dt} = 1000\left[(1+t^2)^{-2} \cdot 2t\right]$

$= 1000\left[\frac{2t}{(1+t^2)^2}\right]$

When $t=3$, $\frac{dN}{dt} = 1000\left[\frac{6}{(1+9)^2}\right]$

$= 1000\left[\frac{6}{100}\right]$

$= 60$

Rate three weeks after disease was introduced will be 60 people/week.

a) $54 - 18a + 6a^2 - 2a^3 + \ldots$

(i) Limiting sum exists if $-1 < r < 1$

$r = \frac{-18a}{54}$

$= -\frac{a}{3}$

Let $-1 < -\frac{a}{3} < 1$

$1 > \frac{a}{3} > -1$

$3 > a > -3$

$-3 < a < 3$

Limiting sum exists if $-3 < a < 3$

(ii) $S = \frac{a}{1-r}$

$81 = \frac{54}{1+\frac{a}{3}}$

$81\left(1 + \frac{a}{3}\right) = 54$

$81 + 27a = 54$

$27a = -27$

$a = -1$

b) (i) $A_{\text{smallest circle}} = \pi(5)^2$

$= 25\pi \text{ cm}^2$

$A_{\text{next circle}} = \pi(10)^2$

$= 100\pi \text{ cm}^2$

$A_{\text{largest circle}} = \pi(15)^2$

$= 225\pi \text{ cm}^2$

$P(10 \text{ points}) = \frac{25\pi}{225\pi}$

$= \frac{1}{9}$

(ii) $A_{\text{inner circle}} = \pi(5)^2$

$= 25\pi \text{ cm}^2$

$A_{\text{first annulus}} = \pi(10)^2 - \pi(5)^2$

$= 75\pi \text{ cm}^2$

$A_{\text{second annulus}} = \pi(15)^2 - \pi(10)^2$

$= 125\pi \text{ cm}^2$

$P(10) = \frac{25\pi}{225\pi} = \frac{1}{9}$

$P(5) = \frac{75\pi}{225\pi} = \frac{1}{3}$

$P(1) = \frac{125\pi}{225\pi} = \frac{5}{9}$

$P(\text{Total of } 20) = P(10,5,5) + P(5,10,5) + P(5,5,10)$

$= \left(\frac{1}{9} \times \frac{1}{3} \times \frac{1}{3}\right) \times 3$

$= \frac{1}{27}$

c) (i) $A_1 = 0.99 \times 60000 + 100$

(ii) $A_2 = 0.99\left[0.99 \times 60000 + 100\right] + 100$

$= 0.99^2 \times 60000 + 0.99 \times 100 + 100$

$A_3 = 0.99\left[A_2\right] + 100$

$= 0.99^3 \times 60000 + 0.99^2 \times 100 + 0.99 \times 100 + 100$

$= 0.99^3 \times 60000 + 100\left[0.99^2 + 0.99 + 1\right]$

$\therefore A_n = 0.99^n \times 60000 + \left[1 + 0.99 + 0.99^2 + \ldots + 0.99^{n-1}\right]$

$\therefore A_n = 0.99^n \times 60000 + 100\left[\frac{1(1 - 0.99^n)}{0.01}\right]$

$= 0.99^n \times 60000 + 10000\left[1 - 0.99^n\right]$

$= 60000(0.99)^n + 10000 - 10000(0.99)^n$

$= 50000(0.99)^n + 10000$

(iii) 90% of initial stock $= 0.9 \times 60000$

$= 54000$

Let $50000(0.99)^n + 10000 < 54000$

$50000 \times 0.99^n < 44000$

$0.99^n < \frac{44}{50}$

$0.99^n < 0.88$

$\log 0.99^n < \log 0.88$

$n > \frac{\log 0.88}{\log 0.99}$

$n > 12.7193\ldots$ month

$\therefore$ It will be in the 13th month (i.e. January of 2001)